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Critical indices in three dimensions

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Abstract. A modification of Wilson's ϵ-expansion scaling procedure is adapted to direct calculation in three dimensions. The parameter $\log r$ is replaced by $(r^{-\epsilon/2}-1)$ where $r = t^{\gamma}$ (t = temperature) and all calculations are carried out directly at $\epsilon = 1$. Several conclusions are drawn: (i) Numerical agreement with the best known data for γ and η is excellent (1 % in γ and 10 % in η) whereas the corresponding values are far less good in the ϵ-expansion where η is off by a factor of more than two. (ii) The series appears to converge very well when corrections to the leading term are summed two by two. (iii) Extrapolations of the early orders of the ϵ-expansion to $\epsilon = 1$ are unjustified.

1. Introduction

In this paper we calculate the critical indices by a technique which draws its inspiration from the work of Wilson (1972)‡ on the renormalization group and the ϵ-expansion. The method we use is a Feynman graph expansion in d dimensions, hence more applicable to the physical case $\epsilon = 1$ than Wilson's. The advantages of our method are:

(i) All integrals are carried out in three dimensions.

(ii) The zeroth order approximation contains that part of the calculation which is independent of the dimensional anomaly η. Thus the corrections are small (about 5 %) because η is small. (Unfortunately however we have not succeeded in converting our series into an η-expansion.) Another noteworthy feature of our procedure is that scaling is guaranteed at each stage of approximation. This is not true of the ϵ-expansion when $\epsilon = 0(1)$ as will be shown in the text.

On the numerical side the quantitative success of our calculation is quite remarkable. For the Ising model at the third stage of approximation—to be defined in the text—we calculate $\gamma = 1{\cdot}24$ and $\eta = 0{\cdot}048$ as against the best series values of

$$\gamma = 1{\cdot}250 \pm 0{\cdot}002, \qquad \eta = 0{\cdot}041^{+0{\cdot}006}_{-0{\cdot}003}.$$

Scaling then gives $\nu = \gamma(2-\eta)^{-1} = 0{\cdot}64$ (best series calculation $\nu = 0{\cdot}638^{+0{\cdot}002}_{-0{\cdot}001}$, Jasnow *et al* 1969). As one of us (Brout 1974) has emphasized these scaling type theories are applicable to weakly scaled quantities only and hence insufficient to calculate α and δ. But where they are applicable, our approach which is numerically quite simple seems to supply a convenient reliable method of estimation, albeit still offering no

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‡ For a review see Wilson and Kogut (1972).

explanation for the remarkable adherence of thermodynamic indices α, β, γ, δ to simple rational numbers.

Finally, a word is in order on the convergence of our approximation scheme. As in all current calculations of these types there is no firm grip on the nature of the convergence of our series. However on the basis of the first few terms we deduce that the oscillating character of the series maintains the stability of the lowest-order calculational results to a remarkable precision. One should, nay must, sum the terms two by two. The stability of the lowest-order term against modification by further effects provides the explanation of the smallness of η. The lowest-order term obtains when $\eta = 0$.

The article is organized as follows. Section 2 presents the method, the central idea of which is Wilson's (1972), namely one determines an effective 'bare' potential, u, through the imposition of scaling. We remark parenthetically that this is certainly *not* the original bare interaction appearing in the Lagrangian, but rather as Wilson has explained an effective interaction which arises due to intermediaries of higher momentum states which have been integrated out. Or if one wishes it is an inversion of the four-point amplitude whose scaling properties are given, ie one can always express the four-point Green function in terms of a potential and vice versa. Imposing the scaling condition on one determines the other. Section 2 is divided up into subsections which present, respectively, the general technique, the calculation to $O(u^3)$ and the calculation to $O(u^4)$.

The method is presented in the simplest and most direct way, at the expense of a possible misunderstanding. Namely the scaling condition is imposed in unrenormalized language where terms in powers of the cut-off are retained. This is however an illusory use of the cut-off. To make this point explicit we discuss in appendix 1 the graphs which contribute to the spherical model in terms of renormalized quantities. It is then seen in a simple case how the cut-off drops out in expressions for ratios of quantities characterized by masses or momenta small compared to the cut-off. One can then understand how it can come about that imposition of the scaling condition leads to series which are more convergent than would *a priori* have been expected.

In appendix 2 we compare the scaling properties of the ϵ-expansion with our method (carried out up to $O(\epsilon^2)$). We show that when $\epsilon = 1$ scaling is violated in this order for $n \lesssim 23$ in the ϵ-expansion.

2. Method

2.1. General technique

We work with the scalar field model, ϕ, an n-component field. The bare Lagrangian† is

$$L = \int d^d x \left(\frac{1}{2} |\partial\phi|^2 + \frac{1}{2}\mu_0^2 |\phi|^2 + \frac{g_0}{4!}(|\phi|^2)^2 \right)$$

$$|\phi|^2 = \sum_{\alpha=1}^{n} \phi_\alpha^2 \qquad (2.1)$$

with a lower limit in spatial integrals situated at $X = 0(\Lambda^{-1})$. The single-particle

† For the precise relation between this Lagrangian and statistical models see the review article of Brout (1974, p 10) and the bibliography contained therein.

propagator is designated by $S(q, r)$, q is the momentum

$$S(q, r) = \int d^d x \exp(qx\langle\phi(x)\phi(0)\rangle$$

and r is the 'mass' defined by $S^{-1}(0, r) = r = t^\gamma$, inverse susceptibility.

We shall calculate the four-point scattering amplitude with (amputated) legs of zero momentum $A(r)$ as a function of r as a power series in an effective potential u, to be distinguished from g_0 of (2.1). Following Wilson, we imagine that many preliminary integrations of higher momentum components have been performed so that the momenta in the effective Feynman graphs generated by u run from $0 \leqslant q \leqslant \lambda$ where $\lambda \ll \Lambda$. By convention we shall set λ equal to unity in our formulae. u is determined by the scaling requirement (Wilson 1972, expression 9):

$$A(r) \simeq r^{(\epsilon - 2\eta)/(2-\eta)}. \tag{2.2}$$

In good approximation (about 5 %) one may neglect η in the determination of u and set

$$A(r) \simeq r^{\epsilon/2}. \tag{2.3}$$

As a first approximation we shall determine u from (2.3). This is then used to determine η and if one wishes one may then return to include the effect of η in determining u as the beginning of a recurrence scheme. However the precision of the calculation appears to be of the order of 1 % and at this level such a program would constitute a needless expenditure of time and energy. This is not to say that a careful theoretical study of this problem in exact terms may not be fruitful in bringing to light interesting qualitative information concerned with the conformal group and the simple rational numbers of the thermodynamic indices. Our point here is that in numerical series expansion methods, either ϵ, $1/n$ or the present one, retention of η in (2.2) seems pointless at the present stage of precision.

In addition to the amplitude A we shall calculate two properties of the propagator S, namely $\partial S^{-1}(0, r)/\partial q^2$ and $\partial S^{-1}(0, r)/\partial t$; t is the dimensionless temperature related to r by $S^{-1}(0, r) = t^\gamma = r$. These quantities are needed to evaluate η and γ respectively. All integrals in the Feynman graphs so generated will be given by a series of powers of $(r^{-\epsilon/2} - 1)$. (The mechanism of cut-off is chosen so as to give this simple form.) A given function which scales according to, say, r^{-x} will then be represented according to

$$r^{-x} = \sum_n A_n (r^{-\epsilon/2} - 1)^n. \tag{2.4}$$

The A_n are given by comparison with

$$r^{-x} = [1 + (r^{-\epsilon/2} - 1)]^{2x/\epsilon} = 1 + \frac{2x}{\epsilon}(r^{-\epsilon/2} - 1) + \frac{1}{2!}\frac{2x}{\epsilon}\left(-1 + \frac{2x}{\epsilon}\right)(r^{-\epsilon/2} - 1)^2 + \dots \tag{2.5}$$

It is seen that the exponent x is given by A_1 through

$$x = \frac{\epsilon A_1}{2}. \tag{2.6}$$

As for the convergence of a series of the type (2.5), it would appear *a priori* that these series should converge very slowly since the parameter $(r^{-\epsilon/2} - 1)$ is very large. The point is that the theory is renormalizable and in terms of renormalized theory the parameter r may be interpreted to be r/r' where r' is a subtraction point taken very close to r. The series will then converge rapidly and the powers of r/r' when this ratio is chosen near to unity are all that is required to determine the theory. Thus it is the renormalizability of the theory (and the concomitant re-interpretation of r in terms of r/r') which gives sense to the study of series such as (2.4) in terms of the first few terms even when $r \ll 1$. This point is made clear in terms of the simple example of spherical model graphs in appendix 1.

It should be noted that the expansion in powers of $\log r$ used in the Wilson's ϵ-expansion presents a similar difficulty. If we expand $r^{-\epsilon/2}$ as

$$r^{-\epsilon/2} = 1 - \frac{\epsilon/2}{1!} \lg r + \frac{(\epsilon/2)^2}{2!} \lg^2 r + \dots \tag{2.7}$$

the series oscillates and stabilizes only after $-\frac{1}{2}\epsilon \lg r$ terms. As ϵ is finite,

$$-\tfrac{1}{2}\epsilon \lg r \to \infty$$

when $r \to 0$. The same argument as above, wherein r is to be replaced by r/r', is then necessary to give a sense to this expansion as well.

Let us now characterize the coefficient of the term in $(r^{-\epsilon/2}-1)$ in the various quantities which will be calculated. We write the scaling properties† of A, $\partial S^{-1}/\partial t$ and $\partial S^{-1}/\partial q$, and define a, b and c by the following relations:

$$A(r) = r^{(\epsilon-2\eta)/(2-\eta)} = 1 + c(r^{-\epsilon/2}-1) + \dots \tag{2.8}$$

$$\frac{\partial S^{-1}(0,r)}{\partial t} = t^{\gamma-1} = r^{(\gamma-1)/\gamma} = 1 + a(r^{-\epsilon/2}-1) + \dots \tag{2.9}$$

$$\frac{\partial S^{-1}(0,r)}{\partial q^2} = r^{-\eta/(2-\eta)} = 1 + b(r^{-\epsilon/2}-1) + \dots, \tag{2.10‡}$$

whence relation (2.6) gives a, b and c in terms of the critical indices γ and η:

$$c = -\frac{1-(2\eta/\epsilon)}{1-(\eta/2)} \simeq -1 \tag{2.11}$$

$$a = -\frac{2}{\epsilon}\frac{\gamma-1}{\gamma} \quad \text{or} \quad \gamma = \frac{1}{1+a\epsilon/2} \tag{2.12}$$

$$b = \frac{\eta}{\epsilon(2-\eta)} \simeq \frac{\eta}{2\epsilon} \qquad (\eta \simeq 2\epsilon b). \tag{2.13}$$

Equation (2.11) with the neglect of η is used to determine u. We remark that with $c \simeq -1$, A is in good approximation a geometric series in $(r^{-\epsilon/2}-1)$. The coefficients a and b are then determined in terms of u, and γ and η determined from (2.12) and (2.13).

The expansion in Feynman graphs is given in figure 1. The cut-off procedure is shown in figure 2. Explanations of the graphical notation are contained in the captions. The subtraction procedure in mass and integrating over $0 \leqslant q \leqslant \infty$ rather than calculating integrals from $0 \leqslant q \leqslant 1$ is the trick that facilitates extraction of the factors $(r^{-\epsilon/2}-1)$. (Of course the ultimate results are independent of the cut-off procedure; one chooses that which is most convenient to one's purpose.)

† See Wilson (1972, formula 9) or for more details Brout (1974, chap. 5 (relation 5.1)).

‡ To obtain this relation (2.10) one uses the scaling property of $S^{-1}(q^2, \mu^2) = \mu^2(\mu/\Lambda)^{-\eta} f(q^2/\mu^2)$ (μ is the correlation length, $\mu \propto t^{\gamma}$) together with the existence of a limit for the derivative of $f(q^2/\mu^2)$ when $q^2 \to 0$. For the scaling relation of $S^{-1}(q^2, \mu^2)$ see Brout (1974, chap. 6, § 3).

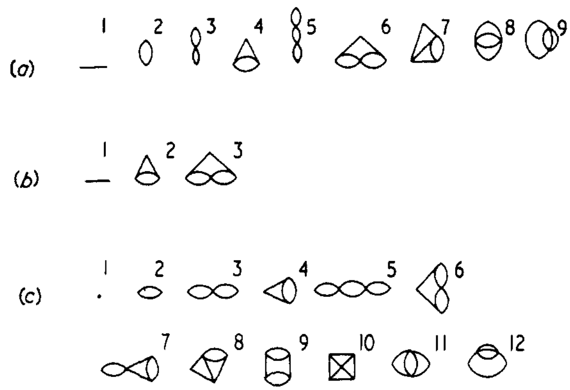

Figure 1. Graphs of the propagator: (*a*) $\partial S^{-1}/\partial t$; (*b*) $\partial S^{-1}/\partial q^2$; and (*c*) the four-point amplitude $A(q^2, r)$. A graph is denoted in the text and the tables by a letter *a*, *b* or *c*, which denotes which quantity ($\partial S^{-1}/\partial t$, $\partial S^{-1}/\partial q^2$, A) is being calculated, as well as by a number which denotes the particular graph which is being calculated.

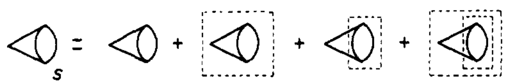

Figure 2. Subtraction procedure for the graph 4(*c*) (figure 1). The mass in a box (dotted lines) is set to unity; each box corresponds to a multiplicative factor of -1.

The calculated values of the various graphs are tabulated in tables 1, 2 and 3. The coefficients of $(r^{-\epsilon/2}-1)$, $(r^{-\epsilon/2}-1)^2$ and $(r^{-\epsilon/2}-1)^3$ have been separated.

2.2. *Second-order calculation*

The second-order calculation contains only chains of bubbles, both in $\partial S^{-1}/\partial t$ and A. These give rise to no momentum dependence and η vanishes in this approximation. These graphs (graphs 1(*c*) and 2(*c*), figure 1) (exact in the spherical model $n \rightarrow \infty$) automatically give rise to a geometric series and consistency requires $c = -1$ (equation (2.11)) or

$$ug_2^{(c)}p_2^{(c)} = -1. \tag{2.14}$$

The notation $g_i^{(\alpha)}$ is the value of a graph calculated at the dimension d (appendix 3) and $p_i^{(\alpha)}$ (appendix 4) is the number of times it appears in a given quantity (i is the graph; α the function that is being calculated). Our convention for the three values of α are:

$$\begin{aligned} \alpha &= a \qquad \partial S^{-1}/\partial t \\ \alpha &= b \qquad \partial S^{-1}/\partial q^2 \\ \alpha &= c \qquad A. \end{aligned} \tag{2.15}$$

In these terms the coefficient a of equations (2.12) and (2.9) together with the $O(u^2)$ expansion of $\partial S^{-1}/\partial t$ (graphs 1(*a*), 2(*a*), figure 1) gives

$$a = ug_2^{(a)}p_2^{(a)}. \tag{2.16}$$

Table 1. Four-point amplitude graphs. The first column defines the graphs according to notations of figure 1. The second column is a list of combinatorial factors (appendix 4). The next columns contain the values of the graphs (appendix 3) for three dimensions and for $d = 4-\epsilon$ dimensions ($\epsilon \to 0$). The last columns contain the coefficients of $(r^{-\epsilon/2}-1)$, $(r^{-\epsilon/2}-1)^2$, $(r^{-\epsilon/2}-1)^3$. For each graph the subtraction procedure depicted by figure 2 is taken into account.

				Contribution[a] to the amplitude expanded in $\Sigma_{\lambda=1}^{\infty} h_\lambda(\epsilon)(r^{-\epsilon/2}-1)^\lambda$		
Graph	Weight	$d = 3$	ϵ-expansion	$h_1(\epsilon)$	$h_2(\epsilon)$	$h_3(\epsilon)$
$c1$	1					
$c2$	$(n+8)/2$	$\pi/4$	$\frac{1}{\epsilon}(1-\epsilon/2+\ldots)$	$ug_2^{(c)}$		
$c3$	$(n^2+6n+20)/4$	$(\pi/4)^2$	$\frac{1}{\epsilon^2}(1-\epsilon+\ldots)$	0	$u^2(g_2^{(c)})^2$	
$c4$	$5n+22$	$\pi^2/24$	$\frac{1}{2\epsilon^2}(1-\epsilon/2+\ldots)$	$u^2[2g_4^{(c)}-(g_2^{(c)})^2]$	$u^2g_4^{(c)}$	
$c5$	$(n^3+8n^2+24n+48)/8$	$(\pi/4)^3$		0	0	$u^3(g_2^{(c)})^3$
$c6$	$(3n^2+22n+56)/2$	0·252		$u^3[3g_6^{(c)}-4g_2^{(c)}g_4^{(c)}+(g_2^{(c)})^3]$	$u^3(3g_6^{(c)}-2g_2^{(c)}g_4^{(c)})$	$u^3g_6^{(c)}$
$c7$	$(3n^2+22n+56)/2$	$\pi^3/96$		0	$u^3g_2^{(c)}[2g_4^{(c)}-(g_2^{(c)})^2]$	$u^3g_2^{(c)}g_4^{(c)}$
$c8$	$2(n^2+20n+60)$	0·115		$u^3\{3g_8^{(c)}-g_2^{(c)}[2g_4^{(c)}-(g_2^{(c)})^2]\}$	$u^3(3g_8^{(c)}-g_2^{(c)}g_4^{(c)})$	$u^3g_8^{(c)}$
$c9$	$(n^2+20n+60)/2$	0·252		$u^3\{3g_9^{(c)}-g_2^{(c)}[4g_4^{(c)}-(g_2^{(c)})^2]\}$	$u^3(3g_9^{(c)}-2g_2^{(c)}g_4^{(c)})$	$u^3g_9^{(c)}$
$c10$	$3(5n+22)$	$\sim$0·05		$3u^3g_{10}^{(c)}$	$3u^3g_{10}^{(c)}$	$u^3g_{10}^{(c)}$
$c11$	$(3n^2+22n+56)/4$	$\pi^3/128$		$3u^3g_{11}^{(c)}$	$u^3[3g_{11}^{(c)}-(g_2^{(c)})^3]$	$u^3g_{11}^{(c)}$
$c12$	$(n^2+10n+16)/2$	neglected				

a The combinatorial factor has been omitted for simplicity; it is found in column 2.

b For $r \neq 1$, these values are multiplied by a factor $(r^{-\epsilon/2})^\lambda$ where λ is equal to the number of integration loops in the graph. (As an example, graph $c2$ is $(\pi/4)r^{-1/2}$ at $d = 3$.)

Table 2. Graphs of $\partial S^{-1}/\partial t$. In column 1, the graphs are defined in the series for $\partial S^{-1}/\partial t$ according to the notation of figure 1. Expressions in the ϵ-expansion, or values at $d = 3$ (appendix 3), are the same as for the corresponding graph in the amplitude as it appears in column 3. The proper weight factors (appendix 4) are taken from column 2.

Graph	Weight	For contribution see table 1
$a1$	1	
$a2$	$(n+2)/2$	$c2$
$a3$	$(n+2)^2/4$	$c3$
$a4$	$3(n+2)/4$	$c4$
$a5$	$(n+2)^3/8$	$c5$
$a6$	$(n+2)(n+8)/4$	$c6$
$a7$	$(n+2)(n+8)$	$c8$
$a8$	$9(n+2)^2/4$	$c11$
$a9$	$(n+2)^2/2$	$c12$

Table 3. Graphs of $\partial S^{-1}/\partial q^2$. Column 1 defines the graph according to the notation in figure 1 In columns 3 and 4 are the values at $d = 3$ (appendix 3) and in the ϵ-expansion respectively. In the three last columns, the contributions to $(r^{-\epsilon/2} - 1)$, $(r^{-\epsilon/2} - 1)^2$, $(r^{-\epsilon/2} - 1)^3$ are given The subtraction procedure of figure 2 has been taken into account In these contributions the combinatorial factors (appendix 4) have been omitted for simplicity They are found in column 2 for $r \neq 1$, see note (b) of table 1.

				Contribution to $\partial S^{-1}/\partial q^2$ expanded in $\Sigma_{\lambda=1}^{3} h_\lambda(\epsilon)(r^{-\epsilon/2} - 1)^\lambda$		
Graph	Weight	Value at $d = 3$	ϵ-expansion	$h_1(\epsilon)$	$h_2(\epsilon)$	$h_3(\epsilon)$
$b1$	1	1				
$b2$	$(n+2)/2$	$-5\pi^2/216$	$-\frac{1}{8\epsilon}(1-\epsilon/4+O(\epsilon^2))$	$2u^2 g_2^{(b)}$	$u^2 g_2^{(b)}$	0
$b3$	$(n+2)(n+8)/4$	$-0\,149$	$-\frac{1}{6\epsilon^2}(1-\epsilon/4+O(\epsilon^2))$	$u^3(3g_3^{(b)} - 4g_2^{(c)}g_2^{(b)})$	$u^3(3g_3^{(b)} - 2g_2^{(c)}g_2^{(b)})$	$u^3 g_3^{(b)}$

Equations (2.16), (2.14) and (2.12) give

$$\gamma = \left(1 - \frac{\epsilon}{2}\frac{p_2^{(a)}}{p_2^{(c)}}\right)^{-1} = \left(1 - \frac{\epsilon}{2}\frac{n+2}{n+8}\right)^{-1},$$

where the second term of (2.17) is obtained by reading off the weights from tables 1 and 2.

In this simple approximation the role of dimensionality in determining γ is restricted to the explicit ϵ dependence displayed in (2.17) and does not manifest itself through the dimensional dependence of the integrals corresponding to the graphs. In three dimensions the value of γ is thus a pure combinatorial problem. One finds $\gamma = 1{\cdot}14$, 1·20, 1·29 for $n = 0, 1, 3$ respectively as against the best calculated values of 7/6, 5/4 and 1·40 (Domb 1970). The deviations are in the range 2–8 %.

The value of u determined from (2.14) may be used to estimate η, say to $O(u^2)$. To calculate the b of equation (2.13) one expands $\partial S^{-1}/\partial q^2$ using the graph 2(*b*) (figure 1):

$$b = 2g_2^{(h)}p_2^{(h)}u^2 \tag{2.18}$$

which with (2.13) and (2.14) gives

$$\eta = 2\epsilon \frac{p_2^{(h)}g_2^{(h)}}{(p_2^{(c)}g_2^{(c)})^2}. \tag{2.19}$$

This shows that η depends on the way the dimensionality occurs in the integrals. In the case $\epsilon \to 0$, one recovers Wilson's estimate (Wilson 1972, formula 2) by using (2.19) and tables 1 and 3:

$$\eta = \frac{1}{2}\frac{n+2}{(n+8)^2}\epsilon^2 + O(\epsilon^3) \tag{2.20}$$

whereas direct evaluation at $\epsilon = 1$ ($d = 3$) of equation (2.19) gives

$$\eta = \frac{40}{27}\frac{n+2}{(n+8)^2} + \text{terms in } O(u^3) \text{ and higher.} \tag{2.21}$$

The factor of about three between equations (2.21) and (2.20) is just what is needed to bring the value of η into line. Clearly direct evaluation at $d = 3$ appears to be a better way to go about things. The detailed numbers are reported in table 4.

Finally in table 5 we show a comparison of the higher-order terms in $(r^{-\epsilon/2}-1)$ when calculated on the one hand with the lowest-order estimate (2.14) and on the other hand self-consistently to $O(u^3)$. It will be seen that the crude estimate satisfies the exact scaling requirement tolerably well. It is then not surprising that the calculated indices are not far out either.

2.3. *Calculation to $O(u^3)$ and $O(u^4)$*

One might expect that the success of the lowest-order calculation would give rise to but small corrections in the higher-order terms. In point of fact this expectation is not borne out when the calculation is carried out to $O(u^3)$ owing to an interesting circumstance. Namely it gives a complex value for the coupling u (for $\epsilon = 1$) when $n \lesssim 15$. The scaling requirement is a second-order algebraic equation in $O(u^3)$. Thus in $O(u^3)$ there exists no real negative coupling u in the range $n \lesssim 15$, ie corresponding to the required repulsive interaction. This is shown in figure 3. It is rather surprising that such a circumstance should arise in the evaluation directly carried out at $d = 3$ whereas similar difficulties do not seem to arise in the ϵ-expansion. However this ability of the ϵ-expansion to scale up to $O(\epsilon^3)$ is but illusory when the ϵ-expansion is examined at $\epsilon = 1$. This point is taken up in appendix 2 where it is shown that it is not possible to find a real negative u to $O(\epsilon^2)$ for $\epsilon = 1$ and $n \lesssim 23$. It appears that use of the ϵ-expansion at small ϵ to determine physical properties at $\epsilon = 1$ is unjustified, at least in the perturbation sense currently in use.

Fortunately, the situation changes radically at the order of u^4. There is very considerable compensation between the effects of $O(u^3)$ and $O(u^4)$—in fact to such an extent that the lowest-order condition for scaling $ug_2^{(c)}p_2^{(c)} = -1$ is correct to within 20% for all values of n (figure 3). The important point is that the series alternates. The

Table 4. Numerical results. Numerical values for the critical indices of the SAW (self-avoiding walk, $n = 0$), the Ising model ($n = 1$), the planar Heisenberg model ($n = 2$) and the Heisenberg model ($n = 3$) are summarized in this table. The successive columns contain respectively the best numerical values obtained from high-temperature expansions, our calculations at $d = 3$ both to $O(u^2)$ and $O(u^4)$ and finally the values obtained through use of the expansion extrapolated to $\epsilon = 1$. In the latter case the three values of γ for each n come from expansion of γ, of $1/\gamma$ (Nickel 1974), and an expression from Brézin *et al* (1973) (where γ is expressed in terms of the expansion γ_4 and η) The values of γ differ due to the extrapolation to $\epsilon = 1$ of equivalent formulae when $\epsilon \to 0$.

Model	n	High-temperature expansions[a]		$d = 3$ calculation				ϵ-expansion extrapolated to $\epsilon = 1$					
		γ	η	γ		η		γ			η		
				$O(u^2)$	$O(u^4)$	$O(u^2)$	$O(u^4)$	$O(\epsilon)$	$O(\epsilon^2)$	$O(\epsilon^3)$	$O(\epsilon^2)$	$O(\epsilon^3)$	$O(\epsilon^4)$
SAW[b]	0	7/6	0.055[e]	1.14	1.16	0.045	0.039	1.143	1.191	1.106	0.016	0.032	0.024
								1.125	1.175	1.222			
								1.144	1.194	1.106			
Ising	1	5/4	$0.041^{+0.006}_{-0.002}$[f]	1.20	1.24	0.053	0.047	1.200	1.276	1.171	0.018	0.037	0.029
								1.166	1.244	1.195			
								1.202	1.282	1.172			
Planar Heisenberg	2	21/16		1.25	1.33	0.058	0.052	1.250	1.351	1.236	0.020	0.039	0.031
								1.200	1.300	1.263			
								1.253	1.361	1.238			
Heisenberg	3	11/8[d,a]	0.043 ± 0.014[d]	1.29	1.41	0.059	0.055	1.294	1.418	1.300	0.021	0.039	0.032
		$1.375^{+0.02}_{-0.01}$						1.227	1.346	1.325			
								1.298	1.430	1.302			
Spherical model[c]	∞	2	0	2	2	0	0	1.500	1.750	1.875	0	0	0

a For a general review see Domb (1970).
b For the relation between $n = 0$ and the SAW see de Gennes (1972).
c Stanley (1968) relates the limit $n \to \infty$ and the spherical model.
d Value found in the paper of Ritchie and Fischer (1972).
e The value of η has been deduced from $(2-\eta)\nu = \gamma$, with $\gamma = 7/6$ and $\nu = 3/5$ (Watts 1974, Martins and Watts 1971).
f Value of Jasnow *et al* (1969).

Table 5. Check of scaling laws. The exponents of the series $\partial S^{-1}/\partial t$, $\partial S^{-1}/\partial q^2$ and the four-point amplitude, computed with $O(u^2)$ graphs, fix the values for the coefficients of terms in $(r^{-\epsilon/2}-1)^2$ and $(r^{-\epsilon/2}-1)^3$. These values appear in columns 2 and 4. They must be compared with the corresponding values appearing in columns 3 and 5 which have been calculated self-consistently to the stated orders using the data tabulated in tables 1 and 2. Agreement is semiquantitative. $\partial S^{-1}/\partial q^2$ has not been tabulated since the order of the calculation is still too small.

Coefficient of	$\partial S^{-1}/\partial t$		four-point amplitude	
$(r^{-\epsilon/2}-1)^2$	0·274	0·259	1	1·22
$(r^{-\epsilon/2}-1)^3$	−0·218	−0·257	−1	−1·49

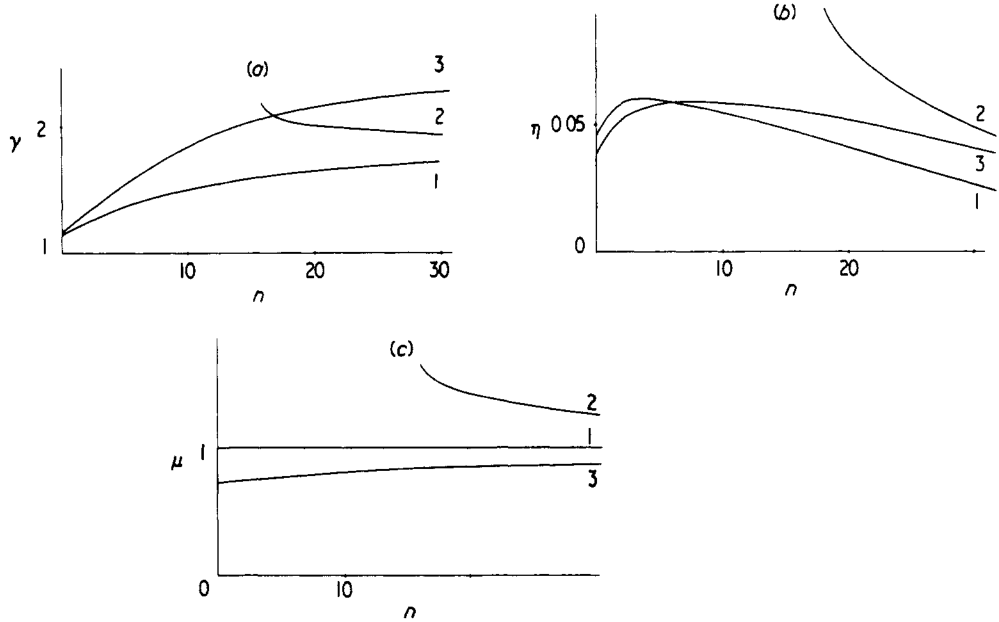

Figure 3. (*a*) index γ; (*b*) index η and (*c*) coupling u in terms of the number of components of the field. For convenience (see point (ii) of the caption) we plot $ug_2^{(c)}p_2^{(c)}$ instead of u. The labels 1, 2, 3 of the curves mean the calculations have been carried out to $O(u^2)$, $O(u^3)$ and $O(u^4)$ respectively. This figure exhibits two main features of the $d = 3$ calculation.

(i) The curve 2 ($O(u^3)$) shows that that the coupling u goes to infinity when $n \to \sim 15$. Below $n \sim 15$ the scaling condition, which is in this case a second degree algebraic equation, gives complex values for u which are physically meaningless. Hence there are no curves (to $O(u^3)$) in the region $0 \lesssim n \lesssim 15$.

(ii) The curves 1 and 3 ($O(u^2)$ and $O(u^4)$) lie very close to each other. This is due to a partial compensation of terms in $O(u^3)$ and $O(u^4)$. This means that the condition $ug_2^{(c)}p_2^{(c)} = -1$ is approximately satisfied in $O(u^4)$ for all n. This shows the convenience of $ug_2^{(c)}p_2^{(c)}$ as variable instead of u.

numerical values for the indices reported in table 4 are in excellent agreement with the best known values.

The stability of the lowest-order calculation against corrections seems to us a good indication of the validity of the method. It is then satisfying that such good agreement with the best series values is obtained.

Acknowledgment

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Appendix 1. Spherical model

In order to illustrate the meaning of the power series of the expansion (2.5), let us consider for the case $n \to \infty$ the ratio of four-point amplitudes for two different masses. After summation of the geometric series of the amplitude one finds

$$A(0, r, \Lambda) = \frac{u}{1-u(g_2^{(c)}(0, r)-g_2^{(c)}(0, \Lambda))}. \tag{A.1.1}$$

The cut-off Λ is of the order of the lattice spacing. Introducing a second mass r', the amplitude can be written as

$$A(0, r, \Lambda) = \frac{u}{1-u(g_2^{(c)}(0, r')-g_2^{(c)}(0, \Lambda))-u(g_2^{(c)}(0, r)-g_2^{(c)}(0, r'))} \tag{A.1.2}$$

or, when $A(0, r', \Lambda)$ is used,

$$\frac{A(0, r, \Lambda)}{A(0, r', \Lambda)} = \frac{1}{1-uA(0, r', \Lambda)[g_2^{(c)}(0, r)-g_2^{(c)}(0, r')]}. \tag{A.1.3}$$

Redefining the coupling u as $uA(0, r', \Lambda)$, the last expression is similar to (A.1.1), where the cut-off Λ has been replaced by the mass r'. But as the choice of r' is free, r' can be taken arbitrarily near to the mass, then each term of the geometric series is as small as is required to get convergence. A similar re-interpretation of Λ in terms of a second mass applies to the original ϵ-expansion in powers of $\lg r$. Moreover, the scaling relation $u'g_2^{(c)}(0, r') = -1$ gives the ratio for the amplitudes:

$$\frac{A(0, r, \Lambda)}{A(0, r', \Lambda)} = \frac{g_2^{(c)}(0, r')}{g_2^{(c)}(0, r)}. \tag{A.1.4}$$

The above mentioned scaling relation $u'g_2^{(c)}(0, r') = -1$ is concerned with the same graphs as those of $O(u^2)$ for arbitrary n (taking into account the proper combinatorial factors). Moreover the absence of infrared divergences for this class allows one to find a similar scaling condition for $q^2 \gg r$.

For the sake of completeness, let us sketch the derivation of the critical indices for the spherical model $n \to \infty$. Recall that η vanishes since the weighting factors in the series $\partial S^{-1}/\partial q^2$ are one order lower in n than the chains of bubbles of the amplitude (η is $O(1/n)$). If the series $\partial S^{-1}/\partial t$ which gives the index γ contains the same graphs as the amplitude, then the scaling exponents are equal for both series. Thus $a = c = -1$ and (equation (2.12)) gives $\gamma = 2$ at $\epsilon = 1$.

Appendix 2. ϵ-expansion

In order to make clear the comparison between the calculation in $d = 3$ to $O(u^3)$ and the ϵ-expansion, we sketch the ϵ-expansion in a manner analogous to the methods of the present paper.

The relation (2.11) for the amplitude gives the following condition when η is neglected:

$$-1 = ug_2^{(c)}p_2^{(c)} + u^2 p_4^{(c)}[2g_4^{(c)} - (g_2^{(c)})^2] + \mathrm{O}(u^3). \tag{A.2.1}$$

Inverting (A.2.1) and introducing $ug_2^{(c)}p_2^{(c)} = -1 + \mathrm{O}(\epsilon)$ one finds

$$\frac{1}{ug_2^{(c)}p_2^{(c)}} = -1 + \frac{p_4^{(c)}}{(p_2^{(c)})^2}\left(2\frac{g_4^{(c)}}{(g_2^{(c)})^2} - 1\right) + \mathrm{O}(\epsilon^2). \tag{A.2.2}$$

The ϵ-expansions of the graphs in (A.2.2) are found in table 1. Inserting them into (A.2.2) the following coupling is deduced:

$$u = \frac{-2\epsilon}{n+8}\,\frac{1}{1+\epsilon/2}\left(1 + 2\frac{5n+22}{(n+8)^2}\epsilon + \mathrm{O}(\epsilon^2)\right). \tag{A.2.3}$$

This result slightly differs from Wilson's (1972)†, due to the approximation in the exponent of the amplitude, where the effects of η are here neglected. If we do not use $\eta = 0$, but $\eta = \frac{1}{2}(n+2)(n+8)^{-2}\epsilon^2$ found previously in (2.20), then complete agreement is obtained:

$$u = \frac{-2\epsilon}{n+8}\,\frac{1}{1+\epsilon/2}\left(1 + \frac{9n+42}{(n+8)^2}\epsilon + \mathrm{O}(\epsilon^2)\right). \tag{A.2.4}$$

But this correction of about 5% is of secondary importance, when $d = 3$. The main difficulties arise from the use of the ϵ-expansion for $\epsilon = 1$ and for physical values of $n = 0, 1, 2, 3$. The derivation contains two sources of errors when ϵ is finite, namely the extrapolations to $\epsilon = 1$ of the values of the graphs, evaluated from the first few terms of their series as well as the corresponding extrapolation of the coupling u.

It is the extrapolation procedure in u which leads to qualitatively misleading results. Namely replacement of the graphical integrals in (A.2.1) by ϵ-expansions does not lead to results which differ by very much from the value of u obtained by calculating the graphs directly in three dimensions. A far more serious error is induced when one inverts (A.2.1) to obtain u as an ϵ-expansion which is truncated at early orders. Effectively, the equation (A.2.1) for the coupling u is soluble only when $n \gtrsim 23$ instead of $n \gtrsim 15$ in the case $d = 3$.

The index γ is related to the series $\partial S^{-1}/\partial t$ by the relation (2.12). One finds:

$$\gamma = \frac{1}{1 + \frac{1}{2}\epsilon u\{p_2^{(a)}g_2^{(a)} + up_4^{(a)}[2g_4^{(a)} - (g_2^{(a)})^2] + \ldots\}}. \tag{A.2.5}$$

The expression (A.2.4) for u and the ϵ-expansions of graphs, found in tables 1 and 2, give

$$\gamma = 1 + \frac{n+2}{2(n+8)}\epsilon + \frac{n+2}{4(n+8)^3}(n^2 + 22n + 52)\epsilon^2 + \mathrm{O}(\epsilon^3). \tag{A.2.6}$$

The index η is obtained from relation (2.13) and the series $\partial S^{-1}/\partial q^2$ (table 3):

$$\eta = \frac{n+2}{2(n+8)^2}\epsilon^2\left(1 + \frac{9n+42}{(n+8)^2}2\epsilon - \frac{\epsilon}{4}\right) + \mathrm{O}(\epsilon^4). \tag{A.2.7}$$

These calculations show how the ϵ-expansion and the procedure in $d = 3$ differ.

† Deduced from equations (8), (9) and (10) of Wilson (1972).

For comparison the values of two typical graphs are listed in both cases, ϵ-expansion and $d = 3$, in table 6. The second source of error in the ϵ-expansion which we have mentioned is more serious. Namely the extrapolation procedure in u allows a coupling to be found although the scaling is not fulfilled in the sense of relation (2.11).

Table 6. Comparison of ϵ-expansion and $d = 3$ calculation for graphs $2c$ and $4c$ (figure 1). The second column is the value at $d = 3$, the third column contains the contribution of the highest pole in ϵ, and the fourth column an evaluation to the next order. Formulae are given in table 1.

Graph	$d = 3$	ϵ-expansion at $\epsilon = 1$	
$g_2^{(c)}$	$\pi/4$ ($\sim$0·785)	1	0·5
$g_4^{(c)}$	$\pi^2/24$ ($\sim$0·411)	0·5	0·25

Appendix 3. Graphs for $d = 3$

We use the following convention to define k_3:

$$k_3 p^2 \, dp = \frac{d^3 p}{(2\pi)^3} \tag{A.3.1}$$

or

$$k_3 = (2\pi^2)^{-1}.$$

Then after integration on the momentum the graph $g_2^{(c)}$ is given by

$$g_2^{(c)} = \frac{k_3 \pi}{2} (q^2)^{-1/2} \tan^{-1} \left(\frac{q^2}{4r} \right)^{1/2}. \tag{A.3.2}$$

The graph $g_4^{(c)}$ is integrated. Taking into account the expression (A.3.2) one finds:

$$g_4^{(c)} = \frac{k_3^2 \pi}{2} \int_0^\infty \frac{p \, \mathrm{d}p}{(p^2 + r)^2} \tan^{-1} \left(\frac{p^2}{4r} \right)^{1/2} = \frac{k_3^2 \pi^2}{24}. \tag{A.3.3}$$

More generally the graphs with exchanges of λ bubbles can be written as:

$$g_\lambda^{(c)} = k_3^{\lambda+1} \left(\frac{\pi}{2} \right)^\lambda \int_0^\infty \mathrm{d}p \frac{[\tan^{-1}(p^2/4r)^{1/2}]^\lambda}{(p^2 + r)^2 (p^2)^{(\lambda-2)/2}}. \tag{A.3.4}$$

The graph $g_8^{(c)}$ taking into account (A.3.2) is written as

$$g_8^{(c)} = \frac{k_3^3 \pi}{2} \int_0^\infty \mathrm{d}p \, \mathrm{d}l \frac{\mathrm{d}(\cos \theta) \frac{1}{2} p^2 l \tan^{-1}(l^2/4r)^{1/2}}{(p^2 + r)^2 [(p + l)^2 + r](l^2 + r)}. \tag{A.3.5}$$

Integrating on $\cos \theta$, it follows that

$$g_8^{(c)} = \frac{k_3^3 \pi}{8} \int_0^\infty \frac{p \, \mathrm{d}p}{(p^2 + r)^2 (l^2 + r)} \tan^{-1} \left(\frac{l^2}{4r} \right)^{1/2} \lg \left| \frac{(p + l)^2 + r}{(p - l)^2 + r} \right|. \tag{A.3.6}$$

An integration by part for the momentum p gives

$$g_8^{(c)} = \frac{k_3^3 \pi^2}{8} \frac{1}{\sqrt{r}} \int_0^\infty \frac{\mathrm{d}l \tan^{-1}(l^2/4r)^{1/2}}{(l^2 + r)(l^2 + 4r)}. \tag{A.3.7}$$

The graph $g_{11}^{(c)}$ taking into account (A.3.2) is written as

$$g_{11}^{(c)} = \frac{k_3\pi}{2}\int_0^\infty \frac{\mathrm{d}p\,\mathrm{d}l\,\frac{1}{2}\mathrm{d}(\cos\theta)p^2 l}{[(p+l)^2+r]^2(p^2+r)^2}\tan^{-1}\left(\frac{l^2}{4r}\right)^{1/2}. \tag{A.3.8}$$

Integrating on the angles, one gets

$$g_{11}^{(c)} = \frac{k_3\pi}{2}\int_0^\infty \frac{\mathrm{d}p\,\mathrm{d}l\,p^2 l\tan^{-1}(l^2/4r)^{1/2}}{(p^2+r)^2[(p+l)^2+r][(p-l)^2+r]} \tag{A.3.9}$$

$$g_{11}^{(c)} = \frac{k_3\pi}{4\sqrt{r}}\int_0^\infty \frac{l\,\mathrm{d}l}{(l^2+4r)^2}\tan^{-1}\left(\frac{l^2}{4r}\right)^{1/2} = \frac{k_3^3\pi^3}{128}r^{-3/2}. \tag{A.3.10}$$

For the graph $g_{10}^{(c)}$ we have not performed an exact calculation. But, as it contains one more momentum than $g_8^{(c)}$, it may be estimated smaller than ($\lesssim 0{\cdot}10$). This gives a value of say 0·05, a crude confirmation of which has been given by a small Monte Carlo calculation.

Graphs of $\partial S^{-1}/\partial q^2$. Taking into account (A.3.2) one finds for $g_2^{(b)}$:

$$g_2^{(b)} = \frac{\partial}{\partial q^2}\bigg|_{q^2=0}\int_i \frac{\mathrm{d}^3 p}{(2\pi)^3}\,\frac{g_2^{(c)}(p^2, r)}{(p+q)^2+r} = -\frac{5\pi^2}{216}. \tag{A.3.11}$$

Similar expressions are found for $g_3^{(b)}$.

Appendix 4. Combinatorial factors

For the sake of completeness we sketch here the calculation of the combinatorial factors (listed in tables 1, 2 and 3) of the Feynman graph expansion of the Lagrangian (2.1). The weight of each graph (figure 1) includes (*a*) a symmetry factor; (*b*) a number of ways of coupling. For point (*a*) the reader is referred to Englert (1961). We specify point (*b*).

The pairing of the indices of the external fields, implied by an interaction term $u(|\phi|^2)^2$ of the Lagrangian, can be depicted with a fictitious heavy meson (dotted line) (figure 4). The three channels (figure 4) are present if one component of the field is concerned. Alternatively, when the components of the external field are different, only one channel occurs.

In general it is necessary to take into account that the n components of the field contribute to coupling when the indices of a subgraph are not connected to other subgraphs. In the language of the fictitious heavy meson, it means that the solid lines of a subgraph are disconnected from the rest of the graph. One has then to classify the graphs with completely decomposed vertices according to their number l of unconnected solid lines; the weight of the class is sn^l, where s denotes the number of graphs of the class l.

Moreover, it is always possible to express the weight of a graph of order p by means of the weights of graphs of order $p-1$. The complete decomposition of the graph of order $p-1$ is worked out by decomposing one of the vertices (figure 4). If this operation disconnects two solid lines, the weight of the corresponding $p-1$ graph has to be multiplied by n; in other cases it has to be multiplied by one. The mechanism of this procedure appears in figure 4 for the first orders.

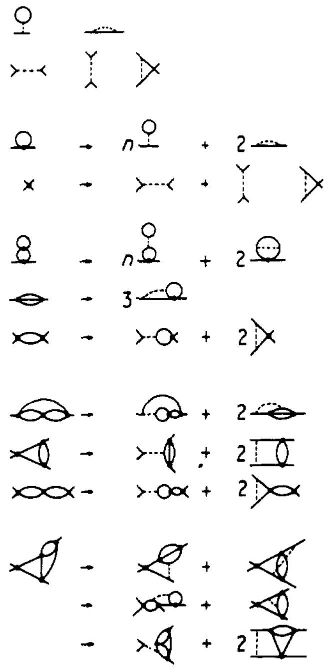

Figure 4. Weight factors. Relation between weight factors of graphs of successive orders. Results are listed in tables 1, 2, 3. As an example, the weight factor of graph 4*c* (figure 1(*c*)) is the weight of graph 4*a* (figure 1(*a*)) plus twice the weight of graph 2*c* (figure 1(*c*)). The three alternative ways of decomposing the graph 8*c* (figure 1(*c*)) are given.

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